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COINCIDENCE AND FIXED POINTS FOR SUBSEQUENTIALLY CONTINUOUS MAPS

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Abstract

We introduce the notions of F-subweak commutativity and F-subsequential continuity for a pair (f, F) of single and multivalued maps and using these notions, we give some new coincidence and fixed point theorems under contractive and strict contractive conditions. These results extend previous ones especially the recent results given by Kamran [8] and [9] and also Liu, Wu and Li [11].

Key words and phrases: Hybrid pairs of maps, Subsequentially (resp. F-subsequentially, f-subsequentially) continuous maps, F-subweakly commuting maps.

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1 Introduction and basic preliminaries

Let $(\mathcal{X}, d)$ be a metric space. We denote by $\mathcal{P}_f(\mathcal{X})$ the family of all nonempty and closed subsets of $\mathcal{X}$, by $\mathcal{P}_{fb}(\mathcal{X})$ the collection of all nonempty closed and bounded subsets of $\mathcal{X}$ and by H the Hausdorff metric induced by d on $\mathcal{P}_{fb}(\mathcal{X})$.

The research of common fixed points in the setting of single-valued maps and single and multivalued maps was investigated by many authors from the last years. In 1982, Sessa [14] introduced the concept of weak commutativity maps for single-valued maps. In 1986 and in order to generalize the concepts of commuting and weak commuting maps, Jungck [5] defined the notion of compatible maps. After three years, Kaneko and Sessa [10] extended the notion of compatible maps to the setting of single and multivalued maps as follows:

1.1 Definition [10] *Maps $F : \mathcal{X} \to \mathcal{P}_{fb}(\mathcal{X})$ and $f : \mathcal{X} \to \mathcal{X}$ are called* **compatible** *if $fFx \in \mathcal{P}_{fb}(\mathcal{X})$ for all $x \in \mathcal{X}$ and $H(Ffx_n, fFx_n) \to 0$, as $n \to \infty$ whenever (x_n) is a sequence in $\mathcal{X}$ such that $Fx_n \to A \in \mathcal{P}_{fb}(\mathcal{X})$ and $fx_n \to t \in A$ as $n \to \infty$.*

Later on, Jungck and Rhoades [6] weakened the notion of compatible single and multivalued maps by introducing the concept of weakly compatible maps.

1.2 Definition [6] *$F : \mathcal{X} \to \mathcal{P}_f(\mathcal{X})$ and $f : \mathcal{X} \to \mathcal{X}$ are* **weakly compatible** *if they commute at their coincidence points; i.e. $\{x \in \mathcal{X} : fx \in Fx\} \subset \{x \in \mathcal{X} : fFx = Ffx\}$.*

In a paper submitted before 2006 but published only in 2008, Al-Thagafi and Shahzad [2] weakened the concept of weakly compatible maps by giving the new concept of occasionally weakly compatible maps. Two self-maps f and g of $\mathcal{X}$ are called occasionally weakly compatible maps (shortly owc) if there is a point x in $\mathcal{X}$ such that $fx = gx$ at which f and g commute. This notion is used in 2006 by Jungck and Rhoades [7] to prove some common fixed point theorems. In 2007, Abbas and Rhoades [1] extended the definition of owc maps to the setting of set-valued maps.

1.3 Definition[1] *$f : \mathcal{X} \to \mathcal{X}$ and $F : \mathcal{X} \to \mathcal{P}_{fb}(\mathcal{X})$ are said to be* **occasionally weakly compatible maps** *(shortly owc) if and only if there exists some point x in $\mathcal{X}$ such that $fx \in Fx$ and $fFx \subseteq Ffx$;*
i.e. $\{x \in \mathcal{X} : fx \in Fx\} \bigcap \{x \in \mathcal{X} : fFx \subset Ffx\} \neq \emptyset$.

1.4 Remark If the set $C(f, F) = \{x \in \mathcal{X} : fx \in Fx\}$ of coincidence points of f and F is empty, the pair (f, F) is trivially weakly compatible; but this situation is without interest for the research of common fixed points. If $C(f, F) \neq \emptyset$, the pair (f, F) is nontrivially weakly compatible and, as with many authors, in the case of two single-valued maps, shortly called weakly compatible.

Following Singh and Mishra [15], we have

1.5 Definition[15] *If $x \in \{x \in \mathcal{X} : fx \in Fx\} \bigcap \{x \in \mathcal{X} : fFx \subset Ffx\}$, $F : \mathcal{X} \to \mathcal{P}_{fb}(\mathcal{X})$ and $f : \mathcal{X} \to \mathcal{X}$ are said to be* **(IT)-commuting at** *$x \in \mathcal{X}$.*

In [8], Kamran further generalizes the notion of (IT)-commutativity for hybrid pairs.

1.6 Definition [8] *Let $F : \mathcal{X} \to \mathcal{P}_{fb}(\mathcal{X})$. $f : \mathcal{X} \to \mathcal{X}$ is said to be* **F-weakly commuting** *at $x \in \mathcal{X}$ if $ffx \in Ffx$.*

In their paper [3], Aamri and El Moutawakil defined Property (EA) for self single-valued maps which contains the class of noncompatible maps introduced by Pant [13].

1.7 Definition [3] *Maps f and $g : \mathcal{X} \to \mathcal{X}$ are said to satisfy* **Property (EA)** *if there exists a sequence (x_n) in $\mathcal{X}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t \in \mathcal{X}$.*

Afterwards, Kamran [8] extended Property (EA) to the setting of single and multivalued maps as follows.

1.8 Definition [8] *Maps* $f : \mathcal{X} \to \mathcal{X}$ *and* $F : \mathcal{X} \to \mathcal{P}_{fb}(\mathcal{X})$ *are said to satisfy Property* (EA) *if there exist a sequence* (x_n) *in* $\mathcal{X}$*, some* $t \in \mathcal{X}$ *and* $A \in \mathcal{P}_{fb}(\mathcal{X})$ *such that* $\lim_{n\to\infty} fx_n = t \in A = \lim_{n\to\infty} Fx_n$.

In 2005, Liu et al. [11] defined new properties called common properties (EA) for four maps which contain properties (EA) introduced by Aamri and El Moutawakil and by Kamran.

1.9 Definition [11]
(1) Let f, g, h *and* $k : \mathcal{X} \to \mathcal{X}$.
The pairs (f, h) *and* (g, k) *are said to satisfy* **Common Property (EA)***(in short* **CEA***) if there exist two sequences* (x_n)*,* (y_n) *in* $\mathcal{X}$ *and some* $t \in \mathcal{X}$ *such that* $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} hx_n = \lim_{n\to\infty} ky_n = t \in \mathcal{X}$.
(2) Let $f, g : \mathcal{X} \to \mathcal{X}$ *and* $S, T : \mathcal{X} \to \mathcal{P}_{fb}(\mathcal{X})$.
The pairs (f, S) *and* (g, T) *are said to satisfy common property* (EA) *if there exist two sequences* (x_n)*,* (y_n) *in* $\mathcal{X}$*, some* $t \in \mathcal{X}$ *and* A, B *in* $\mathcal{P}_{fb}(\mathcal{X})$ *such that* $\lim_{n\to\infty} Sx_n = A$*,* $\lim_{n\to\infty} Ty_n = B$*,* $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = t \in A \cap B$.

On the other hand, Pant [12] introduced the concept of reciprocally continuous maps for pairs of single-valued maps.

1.10 Definition [12] *Let* $(\mathcal{X}, d)$ *be a metric space and let* $f, g : \mathcal{X} \to \mathcal{X}$ *be maps.* f *and* g *are reciprocally continuous if* $\lim_{n\to\infty} fgx_n = ft$ *and* $\lim_{n\to\infty} gfx_n = gt$ *whenever* (x_n) *is a sequence in* $\mathcal{X}$ *such that* $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t \in \mathcal{X}$.

After, in 2002, Singh and Mishra extended the definition of reciprocal continuity to the setting of single and multivalued maps as follows:

1.11 Definition [15] $f : \mathcal{X} \to \mathcal{X}$ *and* $F : \mathcal{X} \to \mathcal{P}_f(\mathcal{X})$ *are* **reciprocally continuous** *on* $\mathcal{X}$ *(resp., at* $t \in \mathcal{X}$*) if and only if* $fFx \in \mathcal{P}_f(\mathcal{X})$ *for each* $x \in \mathcal{X}$ *(resp.,* $fFt \in \mathcal{P}_f(\mathcal{X})$*) and* $\lim_{n\to\infty} fFx_n = fA$*,* $\lim_{n\to\infty} Ffx_n = Ft$ *whenever* (x_n) *is a sequence in* $\mathcal{X}$ *such that* $\lim_{n\to\infty} Fx_n = A \in \mathcal{P}_f(\mathcal{X})$*,* $\lim_{n\to\infty} fx_n = t \in A$.

Our first aim to obtain new results in the present paper is to weaken the notion of F-weak commutativity given by Kamran [8] by introducing the concept of F-subweak commutativity. Also, in our second objective, we will weaken the concept of reciprocal continuity of Singh and Mishra [15] by giving the notion of subsequential continuity in hybrid context.

2 Main results

Now, we introduce the next definitions. We begin by the following concept of F-subweak commutativity which weaken the notion of F-weak commutativity given in [8].

2.1 Definition *Let $F : \mathcal{X} \to \mathcal{P}_{fb}(\mathcal{X})$ and $f : \mathcal{X} \to \mathcal{X}$. f is said to be* **F-subweakly commuting** *iff there exists a sequence (x_n) in $\mathcal{X}$ such that $\lim_{n\to\infty} ffx_n \in A = \lim_{n\to\infty} Ffx_n$.*

2.2 Example Let $\mathcal{X} = [0,\infty[$ with the usual metric. Define $f : \mathcal{X} \to \mathcal{X}$ and $F : \mathcal{X} \to \mathcal{P}_{fb}(\mathcal{X})$ by

$$fx = \begin{cases} x^2 & \text{if } 0 \le x \le 4 \\ 1 & \text{if } 4 < x < \infty, \end{cases} \quad Fx = \begin{cases} \{10000\} & \text{if } 0 \le x \le 4 \\ [0,1] & \text{if } 4 < x < \infty. \end{cases}$$

Consider the sequence $(x_n) = (2 + \dfrac{1}{n})$ for $n = 1, 2, \ldots$. We have

$$fx_n = x_n^2, ffx_n = f(x_n^2) = 1,$$

$$Ffx_n = F(x_n^2) = [0,1] \ni 1,$$

therefore f is F-subweakly commuting.
Note that $ffx \notin Ffx$ for all $x \in \mathcal{X}$.

Now, we give the following notions which weaken the concept of reciprocally continuous maps given by Singh and Mishra [15]. The first one is the corresponding definition of [4] in hybrid context.

2.3 Definition *Maps $f : \mathcal{X} \to \mathcal{X}$ and $F : \mathcal{X} \to \mathcal{P}_{fb}(\mathcal{X})$ are* **subsequentially continuous** *on $\mathcal{X}$ (resp., at $t \in \mathcal{X}$) if and only if $fFx \in \mathcal{P}_{fb}(\mathcal{X})$ for each $x \in \mathcal{X}$ (resp., $fFt \in \mathcal{P}_{fb}(\mathcal{X})$) and there exists a sequence (x_n) in $\mathcal{X}$ such that $\lim_{n\to\infty} fx_n = t \in A = \lim_{n\to\infty} Fx_n$, and $\lim_{n\to\infty} Ffx_n = Ft$; $\lim_{n\to\infty} fFx_n = fA$.*

2.4 Example Let $\mathcal{X} = [0,\infty[$ with the usual metric d. Define $f : \mathcal{X} \to \mathcal{X}$ and $F : \mathcal{X} \to \mathcal{P}_{fb}(\mathcal{X})$ by

$$fx = \begin{cases} 2 - x & \text{if } x < 2 \\ x & \text{if } x \ge 2, \end{cases} \quad Fx = \begin{cases} [2, 2+x] & \text{if } x \le 2 \\ [0,2] & \text{if } x > 2. \end{cases}$$

First, notice that $fFx \in \mathcal{P}_{fb}(\mathcal{X})$ for all $x \in \mathcal{X}$ and f and F are discontinuous at $t = 2$. Consider the sequence $(x_n) = \frac{1}{n}$ for $n = 1, 2, \ldots$. We have

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} (2 - x_n) = 2 = t \in \{2\} = A = \lim_{n\to\infty} Fx_n.$$

Also, we have

$$\lim_{n\to\infty} Ffx_n = \lim_{n\to\infty} F(2 - x_n) = [2,4] = F(t) = F(2),$$

$$\lim_{n\to\infty} fFx_n = \lim_{n\to\infty} f[2, 2 + x_n] = \{2\} = fA = f(2).$$

Therefore f and F are subsequentially continuous.
Now, consider the sequence $(x_n) = 2 + \frac{1}{n}$ for $n = 1, 2, \ldots$. We have

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} x_n = 2 = t \in [0,2] = A = \lim_{n\to\infty} Fx_n.$$

Further, we have

$$Ffx_n = Fx_n = [0,2] \neq F(t) = F(2) = [2,4].$$

Hence, f and F are not reciprocally continuous.

2.5 Definition *Maps $f : \mathcal{X} \to \mathcal{X}$ and $F : \mathcal{X} \to \mathcal{P}_{fb}(\mathcal{X})$ are* F**-subsequentially continuous** *on $\mathcal{X}$ (resp., at $t \in \mathcal{X}$) if and only if $fFx \in \mathcal{P}_{fb}(\mathcal{X})$ for each $x \in \mathcal{X}$ (resp., $fFt \in \mathcal{P}_{fb}(\mathcal{X})$) and there exists a sequence (x_n) in $\mathcal{X}$ such that $\lim_{n\to\infty} fx_n = t \in A = \lim_{n\to\infty} Fx_n$, and $\lim_{n\to\infty} Ffx_n = Ft$.*

2.6 Example Let $\mathcal{X} = [0, \infty[$ be with the usual metric. We define $f : \mathcal{X} \to \mathcal{X}$ and $F : \mathcal{X} \to \mathcal{P}_{fb}(\mathcal{X})$ as follows:

$$fx = \begin{cases} x & \text{if } x < 2 \\ 10 & \text{if } x \geq 2, \end{cases} \quad Fx = \begin{cases} [0,x] & \text{if } x \leq 2 \\ [3,2x] & \text{if } x > 2. \end{cases}$$

It is clear to see that f and F are discontinuous at $t = 2$, and $fFx \in \mathcal{P}_{fb}(\mathcal{X})$ for all $x \in \mathcal{X}$. Define a sequence (x_n) in $\mathcal{X}$ by $x_n = 2 - \frac{1}{n}$ for $n = 1, 2, \ldots$. Then, we have

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} x_n = 2 = t,$$

$$\lim_{n\to\infty} Fx_n = \lim_{n\to\infty} [0, x_n] = [0,2] = A \ni t,$$

$$\lim_{n\to\infty} Ffx_n = \lim_{n\to\infty} Fx_n = [0,2],$$

$$F(t) = F(2) = [0,2] = \lim_{n\to\infty} Ffx_n.$$

Therefore the pair (f, F) is F-subsequentially continuous.
Note that the pair (f, F) is not subsequentially continuous.

2.7 Definition *Maps $f : \mathcal{X} \to \mathcal{X}$ and $F : \mathcal{X} \to \mathcal{P}_{fb}(\mathcal{X})$ are* f**-subsequentially continuous** *on $\mathcal{X}$ (resp., at $t \in \mathcal{X}$) if and only if $fFx \in \mathcal{P}_{fb}(\mathcal{X})$ for each $x \in \mathcal{X}$ (resp., $fFt \in \mathcal{P}_{fb}(\mathcal{X})$) and there exists a sequence (x_n) in $\mathcal{X}$ such that $\lim_{n\to\infty} fx_n = t \in A = \lim_{n\to\infty} Fx_n$, and $\lim_{n\to\infty} fFx_n = fA$.*

2.8 Example Let $\mathcal{X} = [0, \infty[$ with the usual metric and

$$fx = \begin{cases} x & \text{if } x \leq 1 \\ 2 & \text{if } x > 1, \end{cases} \quad Fx = \begin{cases} [0,x] & \text{if } x < 1 \\ \{5\} & \text{if } x = 1 \\ [2, x+1] & \text{if } x > 1. \end{cases}$$

We see that f and F are not continuous at $t = 1$ and $fFx \in \mathcal{P}_{fb}(\mathcal{X})$ for each $x \in \mathcal{X}$. Let (x_n) be a sequence in $\mathcal{X}$ defined by $x_n = 1 - \frac{1}{n}$ for $n = 1, 2, \ldots$. We have

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} x_n = 1 = t,$$

$$\lim_{n\to\infty} Fx_n = [0,1] = A \ni t,$$

and

$$\lim_{n\to\infty} fFx_n = \lim_{n\to\infty} f[0,x_n] = \lim_{n\to\infty}[0,x_n] = [0,1] = f(A) = f[0,1].$$

But,

$$\lim_{n\to\infty} Ffx_n = \lim_{n\to\infty} Fx_n = \lim_{n\to\infty}[0,x_n] = [0,1],$$
$$F(t) = F(1) = \{5\} \neq [0,1].$$

Hence f and F are f-subsequentially continuous but not subsequentially continuous.

2.9 Remark From the definitions it is clear that if f and F are subsequentially continuous then the pair (f,F) is F-subsequentially continuous and f-subsequentially continuous.

Let $f,g : \mathcal{X} \to \mathcal{X}$ and $F,G : \mathcal{X} \to \mathcal{P}_{fb}(\mathcal{X})$. In their paper [11], Liu et al. gave some new common fixed points for the hybrid pairs of single and multivalued maps (f,F) and (g,G) under the following strict hybrid contraction condition

$$\begin{aligned}(1) \qquad & H(Fx,Gy)\\ & < \max\left\{d(fx,gy), \frac{d(fx,Fx)+d(gy,Gy)}{2}, \frac{d(fx,Gy)+d(gy,Fx)}{2}\right\}.\end{aligned}$$

Motivated by Kamran [9], we consider the following type condition

$$\begin{aligned}(2) \qquad & H(Fx,Gy)\\ & < \max\left\{d(fx,gy), \frac{d(fx,Fx)+d(gy,Gy)}{2}, \frac{d(fx,Gy)+d(gy,Fx)}{2}\right\}\\ & +Ld(fx,gy),\end{aligned}$$

where $L \geq 0$.

The third aim of our contribution is to show that the conclusions of Th.3.10 of [8] and Th.2.3, 2.6 and 2.7 of [11] remain valid if we replace strict contractive condition (1) by weaker contractive type condition (2).
The next example given in [9] shows the generality of condition (2).

2.10 Example Let $\mathcal{X} = [1,\infty[$ with the usual metric d. Define $f,g : \mathcal{X} \to \mathcal{X}$ and $F,G : \mathcal{X} \to \mathcal{P}_{fb}(\mathcal{X})$ by $fx = gx = x$, and $Fx = Gx = [1,x]$ for all $x \in \mathcal{X}$. Then,

$$\begin{aligned} H(Fx,Gy) &= |x-y|\\ &\nless |x-y|\\ &= \max\left\{d(fx,gy), \frac{d(fx,Fx)+d(gy,Gy)}{2}, \frac{d(fx,Gy)+d(gy,Fx)}{2}\right\}\end{aligned}$$

for all $x \neq y \in \mathcal{X}$. Hence, f,g,F and G don't satisfy condition (1) but they satisfy condition (2) for any $L > 0$.

2.11 Theorem *Let f, g be two self-maps of the metric space $(\mathcal{X}, d)$ and let F, G be two maps from $\mathcal{X}$ into $\mathcal{P}_{fb}(\mathcal{X})$ such that*
(1) (f, F) and (g, G) satisfy the common property (EA);
(2) for all $x \neq y \in \mathcal{X}$, $L \geq 0$

$$\begin{aligned} H(Fx, Gy) \quad &< \quad \max\left\{d(fx, gy), \frac{d(fx, Fx) + d(gy, Gy)}{2}, \frac{d(fx, Gy) + d(gy, Fx)}{2}\right\} \\ &\quad +Ld(fx, gy). \end{aligned}$$

If fX and gX are closed subsets of $\mathcal{X}$, then
(a) f and F have a coincidence point;
(b) g and G have a coincidence point;
(c) f and F have a common fixed point u provided that, for some (x_n) in $\mathcal{X}$, $\lim_{n\to\infty} ffx_n = \lim_{n\to\infty} fx_n = fu$, f is F-subweakly commuting and the pair (f, F) is F-subsequentially continuous;
(d) g and G have the same common fixed point u provided that, for some (y_n) in $\mathcal{X}$, $\lim_{n\to\infty} ggy_n = \lim_{n\to\infty} gy_n = gu$, g is G-subweakly commuting and the pair (g, G) is G-subsequentially continuous.

Proof
Since the pair (f, F) and (g, G) satisfy the common property (EA), there exist two sequences (x_n), (y_n) in $\mathcal{X}$ and $u \in \mathcal{X}$, $A, B \in \mathcal{P}_{fb}(\mathcal{X})$ such that

$$\lim_{n\to\infty} Fx_n = A, \lim_{n\to\infty} Gy_n = B,$$

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = u \in A \cap B.$$

By virtue of fX and gX being closed, we have $u = fv = gw$ for some $v, w \in X$. First, we claim that $gw \in Gw$. Indeed, inequality (2) implies that

$$\begin{aligned} &H(Fx_n, Gw) \\ < \quad &\max\left\{d(fx_n, gw), \frac{d(fx_n, Fx_n) + d(gw, Gw)}{2}, \frac{d(fx_n, Gw) + d(gw, Fx_n)}{2}\right\} \\ &+Ld(fx_n, gw). \end{aligned}$$

Taking the limit as $n \to \infty$, we obtain

$$H(A, Gw) \leq \frac{1}{2} d(gw, Gw).$$

Since $gw = fv = u \in A$, it follows from the definition of Hausdorff metric that

$$d(gw, Gw) \leq H(A, Gw) \leq \frac{1}{2} d(gw, Gw),$$

which implies that $gw \in Gw$.

Now, suppose that $fv \notin Fv$, by condition (2) again, we get

$$\begin{aligned} & H(Fv, Gy_n) \\ < \quad & \max\left\{d(fv, gy_n), \frac{d(fv, Fv) + d(gy_n, Gy_n)}{2}, \frac{d(fv, Gy_n) + d(gy_n, Fv)}{2}\right\} \\ & + Ld(fv, gy_n). \end{aligned}$$

At infinity we obtain

$$H(Fv, B) \leq \frac{1}{2} d(fv, Fv).$$

Since $fv = gw = u \in B$, it follows from the definition of Hausdorff metric H that

$$d(fv, Fv) \leq H(Fv, B) \leq \frac{1}{2} d(fv, Fv),$$

which implies that $fv \in Fv$.
On the other hand, by virtue of condition (c), we have

$$u = \lim_{n\to\infty} fx_n = fu = \lim_{n\to\infty} ffx_n \in \lim_{n\to\infty} Ffx_n = Fu.$$

Thus $u = fu \in Fu$.
Similarly, $u = gu \in Gu$.

∎

2.12 Corollary *Let f be a self-map of the metric space $(\mathcal{X}, d)$ and let F be a map from $\mathcal{X}$ into $\mathcal{P}_{fb}(\mathcal{X})$ such that*
(1) (f, F) satisfy the common property (EA);
(2) for all $x \neq y \in \mathcal{X}$, $L \geq 0$

$$\begin{aligned} & H(Fx, Fy) \\ < \quad & \max\left\{d(fx, fy), \frac{d(fx, Fx) + d(fy, Fy)}{2}, \frac{d(fx, Fy) + d(fy, Fx)}{2}\right\} \\ & + Ld(fx, fy). \end{aligned}$$

If fX is a closed subset of $\mathcal{X}$, then
(a) f and F have a coincidence point;
(b) f and F have a common fixed point u provided that for some (x_n) in $\mathcal{X}$, $\lim_{n\to\infty} ffx_n = \lim_{n\to\infty} fx_n = fu$, f is F-subweakly commuting and the pair (f, F) is F-subsequentially continuous.

Proof
Let $F = G$ and $f = g$, then the results follow immediately from the previous theorem.

∎

For three maps, we get the following result.

2.13 Corollary *Let f be a self-map of the metric space $(\mathcal{X}, d)$ and let F, G be two maps from $\mathcal{X}$ into $\mathcal{P}_{fb}(\mathcal{X})$ such that*
(1) (f, F) and (f, G) satisfy the common property (EA);
(2) for all $x \neq y \in \mathcal{X}$, $L \geq 0$

$$\begin{aligned} & H(Fx, Gy) \\ < \quad & \max\left\{d(fx, fy), \frac{d(fx, Fx) + d(fy, Gy)}{2}, \frac{d(fx, Gy) + d(fy, Fx)}{2}\right\} \\ & + Ld(fx, fy). \end{aligned}$$

If fX is a closed subset of $\mathcal{X}$, then
(a) f, F and G have a coincidence point;
(b) f, F and G have a common fixed point u provided that f is F-subweakly commuting and the pair (f, F) is F-subsequentially continuous; f is G-subweakly commuting and the pair (f, G) is G-subsequentially continuous and $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} ffx_n = fu$ for some (x_n) in $\mathcal{X}$.

Before giving our next result, we recall the following notion originally defined in [4].

2.14 Definition *[4] $f, g : \mathcal{X} \to \mathcal{X}$ are subsequentially continuous iff there exists a sequence (x_n) in $\mathcal{X}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in \mathcal{X}$ and satisfy $\lim_{n\to\infty} fgx_n = ft$ and $\lim_{n\to\infty} gfx_n = gt$.*

2.15 Corollary *Let f, g, S, and T be four self-maps of the metric space $(\mathcal{X}, d)$ such that*
(1) (f, S) and (g, T) satisfy the common property (EA);
(2) for all $x \neq y \in \mathcal{X}$, $L \geq 0$

$$\begin{aligned} H(Sx, Ty) \quad < \quad & \max\left\{d(fx, gy), \frac{d(fx, Sx) + d(gy, Ty)}{2}, \frac{d(fx, Ty) + d(gy, Sx)}{2}\right\} \\ & + Ld(fx, gy). \end{aligned}$$

If fX and gX are closed subsets of $\mathcal{X}$, then
(a) f and S have a coincidence point;
(b) g and T have a coincidence point;
(c) f and S have a common fixed point u provided that f and S are S-subweakly commuting and subsequentially continuous and $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} ffx_n = fu$ for some (x_n) in $\mathcal{X}$;
(d) g and T have the common fixed point u provided that g and T are T-subweakly commuting and subsequentially continuous and $\lim_{n\to\infty} gy_n = \lim_{n\to\infty} ggy_n = gu$ for some (y_n) in $\mathcal{X}$.

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ upper semi-continuous and $0 < \varphi(t) < t$ for each $t \in]0, \infty[$.

2.16 Theorem *Let f, g be self-maps of the metric space $(\mathcal{X}, d)$ and let $F, G : \mathcal{X} \to \mathcal{P}_{fb}(\mathcal{X})$ be multivalued maps such that*
(1) (f, F) and (g, G) satisfy the common property (EA);
(2) for all $x \neq y \in \mathcal{X}$,

$$H(Fx, Gy) \leq \varphi(\max\{d(fx, gy), d(fx, Fx), d(gy, Gy), d(fx, Gy), d(gy, Fx)\}).$$

If fX and gX are closed subsets of $\mathcal{X}$, then
(a) f and F have a coincidence point;
(b) g and G have a coincidence point;
(c) f and F have a common fixed point u provided that for some (x_n) in $\mathcal{X}$, $\lim_{n\to\infty} ffx_n = \lim_{n\to\infty} fx_n = fu$, f is F-subweakly commuting and (f, F) is F-subsequentially continuous;
(d) g and G have the same common fixed point u provided that for some (y_n) in $\mathcal{X}$, $\lim_{n\to\infty} ggy_n = \lim_{n\to\infty} gy_n = gu$, g is G-subweakly commuting and (g, G) is G-subsequentially continuous.

Proof
Since maps pair (f, F) and (g, G) satisfy common property (EA), as in proof of Theorem 2.11, there exist two sequences (x_n), (y_n) in $\mathcal{X}$ and $u \in \mathcal{X}$, $A, B \in \mathcal{P}_{fb}(\mathcal{X})$ such that

$$\lim_{n\to\infty} Fx_n = A, \lim_{n\to\infty} Gy_n = B,$$
$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = u \in A \cap B.$$

By virtue of fX and gX being closed, we have $u = fv = gw$ for some $v, w \in X$. First, we claim that $gw \in Gw$. Indeed, inequality (2) implies that

$$\begin{aligned} & H(Fx_n, Gw) \\ \leq\ & \varphi(\max\{d(fx_n, gw), d(fx_n, Fx_n), d(gw, Gw), d(fx_n, Gw), d(gw, Fx_n)\}). \end{aligned}$$

At infinity, we obtain

$$H(A, Gw) \leq \varphi(d(gw, Gw)) < d(gw, Gw).$$

Since $gw \in A$, from the definition of Hausdorff metric, it follows that

$$d(gw, Gw) \leq H(A, Gw) < d(gw, Gw),$$

which is a contradiction. Therefore $gw \in Gw$.
Similarly, $fv \in Fv$.
Thus f and F have a coincidence point v, g and G have a coincidence point w. The rest of proof is similar to the argument of the above theorem. ∎

For all $x, y \in \mathcal{X}$, since $\frac{d(fx,Gy)+d(gy,Fx)}{2} \leq \max\{d(fx, Gy), d(gy, Fx)\}$, we present the following theorem.

2.17 Theorem *Let $(\mathcal{X}, d)$ be a metric space. Let $f, g : \mathcal{X} \to \mathcal{X}$ and $F, G : \mathcal{X} \to \mathcal{P}_{fb}(\mathcal{X})$ be single and multivalued maps respectively such that*
(1) the pairs (f, F) and (g, G) satisfy the common property (EA);
(2) for all $x \neq y \in \mathcal{X}$ and $\lambda \in]0, 1[$,

$$H(Fx, Gy) \leq \lambda \max \{d(fx, gy), d(fx, Fx), d(gy, Gy), d(fx, Gy), d(gy, Fx)\}.$$

If fX and gX are closed subsets of $\mathcal{X}$, then
(a) f and F have a coincidence point;
(b) g and G have a coincidence point;
(c) f and F have a common fixed point u provided that f is F-subweakly commuting, (f, F) is F-subsequentially continuous and for some sequence (x_n) in $\mathcal{X}$, $\lim_{n\to\infty} ffx_n = \lim_{n\to\infty} fx_n = fu$;
(d) g and G have the common fixed point u provided that g is G-subweakly commuting, (g, G) is G-subsequentially continuous and for some sequence (y_n) in $\mathcal{X}$, $\lim_{n\to\infty} ggy_n = \lim_{n\to\infty} gy_n = gu$.

Proof
Since (f, F) and (g, G) satisfy the common property (EA), as in proof of Theorem 2.11, there exist $u \in \mathcal{X}$, $A, B \in \mathcal{P}_{fb}(\mathcal{X})$, two sequences (x_n) and (y_n) in $\mathcal{X}$ such that

$$\lim_{n\to\infty} Fx_n = A, \lim_{n\to\infty} Gy_n = B,$$
$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = u \in A \cap B.$$

And since fX and gX are closed subsets, there are two points $v, w \in \mathcal{X}$ such that $u = fv = gw$.
Suppose that $gw \notin Gw$. By inequality (2), we get

$$\begin{aligned} & H(Fx_n, Gw) \\ \leq \; & \lambda \max \{d(fx_n, gw), d(fx_n, Fx_n), d(gw, Gw), d(fx_n, Gw), d(gw, Fx_n)\}. \end{aligned}$$

At infinity, we obtain

$$H(A, Gw) \leq \lambda d(gw, Gw) < d(gw, Gw).$$

Since $gw \in A$, we have

$$d(gw, Gw) \leq H(A, Gw)$$

and therefore

$$d(gw, Gw) \leq H(A, Gw) \leq \lambda d(gw, Gw) < d(gw, Gw)$$

which is a contradiction. Hence $gw \in Gw$.
Similarly, $fv \in Fv$.
The rest of the proof is similar to the argument of the above theorems. ∎

We end our work by giving the last result.

2.18 Theorem *Let $(\mathcal{X}, d)$ be a metric space and let $f, g : \mathcal{X} \to \mathcal{X}$; $F, G : \mathcal{X} \to \mathcal{P}_{fb}(\mathcal{X})$ be maps such that*
(1) the pairs (f, F) and (g, G) satisfy the common property (EA);
(2) for all $x \neq y \in \mathcal{X}$

$$
\begin{aligned}
& H(Fx, Gy) \\
\leq\ & \alpha(d(fx, gy))d(fx, gy) \\
& +\beta(d(fx, gy)) \max\{d(fx, Fx), d(gy, Gy)\} \\
& +\gamma(d(fx, gy)) \max\{d(fx, Gy) + d(gy, Fx), d(fx, Fx) + d(gy, Gy)\}
\end{aligned}
$$

where $\alpha, \beta, \gamma : [0, \infty[\to]0, 1[$ be upper semi-continuous functions and which satisfy

$$\beta(t) + \gamma(t) < 1.$$

If fX and gX are closed subsets of $\mathcal{X}$, then
(a) f and F have a coincidence point;
(b) g and G have a coincidence point;
(c) f and F have a common fixed point u provided that f is F-subweakly commuting, (f, F) is F-subsequentially continuous and for some sequence (x_n) in $\mathcal{X}$, $\lim_{n\to\infty} ffx_n = \lim_{n\to\infty} fx_n = fu$;
(d) g and G have the common fixed point u provided that g is G-subweakly commuting, (g, G) is G-subsequentially continuous and for some sequence (y_n) in $\mathcal{X}$, $\lim_{n\to\infty} ggy_n = \lim_{n\to\infty} gy_n = gu$.

Proof
By virtue of the hypotheses, we have

$$\lim_{n\to\infty} Fx_n = A,\ \lim_{n\to\infty} Gy_n = B,$$

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = u \in A \cap B,$$

where $u = fv = gw$.
If $fv \notin Fv$, we have

$$
\begin{aligned}
& H(Fv, Gy_n) \\
\leq\ & \alpha(d(fv, gy_n))d(fv, gy_n) \\
& +\beta(d(fv, gy_n)) \max\{d(fv, Fv), d(gy_n, Gy_n)\} \\
& +\gamma(d(fv, gy_n)) \max\{d(fv, Gy_n) + d(gy_n, Fv), d(fv, Fv) + d(gy_n, Gy_n)\}.
\end{aligned}
$$

Taking the limit as $n \to \infty$, it yields

$$d(fv, Fv) \leq H(Fv, B) \leq [\beta(0) + \gamma(0)]d(fv, Fv) < d(fv, Fv)$$

This contradiction leads to $fv \in Fv$.

Now, suppose that $d(gw, Gw) > 0$. By using condition (2) we get

$$\begin{aligned} & H(Fx_n, Gw) \\ \leq \; & \alpha(d(fx_n, gw))d(fx_n, gw) \\ & +\beta(d(fx_n, gw)) \max\{d(fx_n, Fx_n), d(gw, Gw)\} \\ & +\gamma(d(fx_n, gw)) \max\{d(fx_n, Gw) + d(gw, Fx_n), d(fx_n, Fx_n) + d(gw, Gw)\}. \end{aligned}$$

The passage to the limit gives

$$d(gw, Gw) \leq H(A, Gw) \leq [\beta(0) + \gamma(0)]d(gw, Gw) < d(gw, Gw)$$

which is a contradiction. Hence $gw \in Gw$.
We prove the rest as in the first result. ■

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